Introduction to Modal Logic

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Course Information

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Course Website:	`https://umd.instructure.com/courses/1353445`

Topics

1. Propositional Modal Logic
2. First-Order Modal Logic
3. Non-Normal Modal Logics
4. Applications: (Dynamic) Epistemic Logic, Epistemic Temporal Logic, Logics of Knowledge and Ability

Setting the stage: Classical logic

Propositional Logic (PL)

- Language: $P \wedge Q$, $P \to (Q \vee \neg R)$, etc.
- Proof-Theory: Natural Deduction, Hilbert-style Deductions, Tableaux, etc.
- Semantics: Truth functions

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First-Order Logic (FOL)

- Language: $x = y$, $\exists x \forall y (P(x) \wedge Q(x, y))$, $\forall x \exists y (F(x) \rightarrow (G(x, y) \wedge \neg R(y)))$, etc.
- Proof-Theory: Natural Deduction, Hilbert-style Deductions, Tableaux, etc.
- Semantics: First-order structures

Notes on propositional and first order logic.

Reasoning with classical logic: pros and cons

Advantages:

- relatively simple syntax and well-understood semantics
- well-developed deductive systems and tools for automated reasoning

Disadvantages:

- cannot adequately represent some aspects natural language
- cannot adequately capture specific modes of reasoning
- *undecidability* of logical consequence and validity (for FOL)

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- Modern modal logic started in the early 1960s with the introduction of relational semantics by Saul Kripke (although see the earlier work by McKinsey and Tarski on logic and topology and Gödel on provability logic).
- There are a wide variety of modal systems, with different interpretations of the modal operators. Modal logic is an important tool in many disciplines: philosophy, computer science, linguistics, economics

The History of Modal Logic

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, Vol. 7, 2006.

P. Balckburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.

R. Ballarin. *Modern Origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

What is a modal?

A modality is any word or phrase that can be applied to a statement S to create a new statement that makes an assertion that *qualifies* the truth of S.

Types of Modal Logics

Alethic logic: Necessary and possible truths.

Temporal logic: Temporal reasoning.

Spatial logics: Reasoning about spatial relations.

Epistemic logics: Reasoning about knowledge.

Doxastic logics: Reasoning about beliefs.

Deontic logics: Reasoning about obligations and permissions.

Types of Modal Logics

Logics of multiagent systems: Reasoning about many agents and their knowledge, beliefs, goals, actions, strategies, etc.

Description logics: Reasoning about ontologies.

Logics of programs: Reasoning about program executions.

Logics of computations: Specification of transition systems.

Provability logic: Reasoning about proofs

Introducing Modal Logic

Modern Modal Logic began with C.I. Lewis' dissatisfaction with the material conditional ($\rightarrow$).

- Irrelevance/non-causality:
  If the Sun is hot, then $2 + 2 = 4$.
- False antecedents:
  If $2 + 2 = 5$ then the Moon is made of cheese.
- Monotonicity:
  If I put sugar in my coffee, then it will taste good. Therefore, if I put sugar and I put oil in my coffee then it will taste good.

Introducing Modal Logic

C.I. Lewis' idea: Interpret 'If A then B' as 'It **must** be the case that A implies B', or 'It is **necessarily** the case that A implies B'

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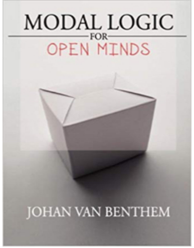
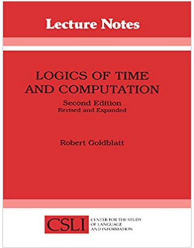
Introducing Modal Logic

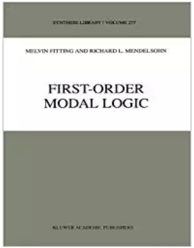
Gradually, the study of the modalities themselves became dominant, with the study of "conditionals" developing into a separate topic.

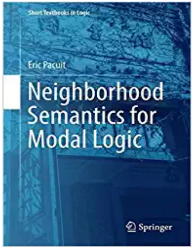
Books

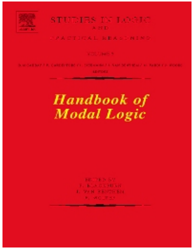

Modal Logic
Patrick Blackburn, Maarten de Rijke and Yde Venema


MODAL LOGIC
FOR
OPEN MINDS
JOHAN VAN BENTHEM

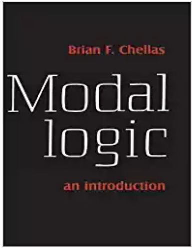

Lecture Notes
LOGICS OF TIME AND COMPUTATION
Second Edition
Revised and Expanded
Robert Goldblatt
CSLI
CENTER FOR THE STUDY OF LANGUAGE AND INFORMATION


SYNTHESE LIBRARY / VOLUME 277
MELVIN FITTING AND RICHARD L. MENDELSOHN
FIRST-ORDER MODAL LOGIC
KLUWER ACADEMIC PUBLISHERS

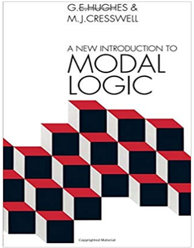

Short Textbooks in Logic
Eric Pacuit
Neighborhood Semantics for Modal Logic
Springer


Handbook of Modal Logic

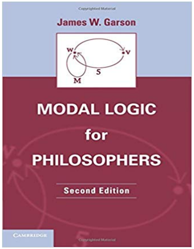
Books



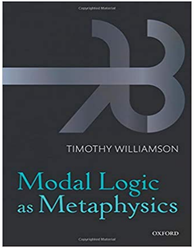
Brian F. Chellas
Modal logic
an introduction



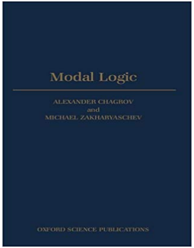
G.E.HUGHES &
M.J.CRESSWELL
A NEW INTRODUCTION TO
MODAL LOGIC



James W. Garson
MODAL LOGIC
for
PHILOSOPHERS
Second Edition



TIMOTHY WILLIAMSON
Modal Logic as Metaphysics
OXFORD



Modal Logic
ALEXANDER CHAGROV
and
MICHAEL ZAKHARYASCHEV
OXFORD SCIENCE PUBLICATIONS

Modal Languages

Modal languages extend some logical language (e.g., propositional logic, first-order logic, second-order logic, etc.) with (at least) two new symbols ‘$\Box$’ and ’$\Diamond$’.

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$\Box\varphi$: "it is *necessary* that φ is true"

$\Diamond\psi$: "it is *possible* that φ is true"

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$\Box\varphi$: "it is *knowing* that φ is true"

$\Diamond\psi$: "it is *consistent with everything that is known* that φ is true"

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$\Box\varphi$: "it is *will always be* that φ is true"

$\Diamond\psi$: "it is *will sometimes be* that φ is true"

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$\Box\varphi$: "it is *ought to be* that φ is true"

$\Diamond\psi$: "it is *permissible* that φ is true"

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$\Box\varphi$: "it is ______ that φ is true"

$\Diamond\psi$: "it is ______ that φ is true"

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An alternative approach treats modals as *predicates* that apply to terms (that are Gödel numbers of sentences)

J. Stern. *Toward Predicate Approaches to Modality*. Springer, 2016.

Narrow vs. Wide Scope

"If you do p, you must also do q"

- $p \rightarrow \Box q$
- $\Box(p \rightarrow q)$

de dicto vs. de re

"I know that someone appreciates me"

- $\Box\exists x A(x, e)$ (*de dicto*)
- $\exists x \Box A(x, e)$ (*de re*)

Iterations of Modal Operators

$\Box\varphi \rightarrow \Box\Box\varphi$: If I know, do I know that I know?

$\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: If I don't know, do I know that I don't know?

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What about: $\Diamond\Box\varphi \to \Box\Diamond\varphi$, $\Box\Diamond\varphi \to \Diamond\Box\varphi$, $\varphi \to \Box\Diamond\varphi$, $\Diamond\Box(\varphi \wedge \psi) \to \Diamond\Box\varphi \wedge \Diamond\Box\psi$, . . .?

Propositional Modal Language

Language: Let At be a set of atomic propositions. The set of propositional modal formulas, denoted $\mathcal{L}(\mathsf{At})$, is the smallest set of formulas generated by the following grammar:

$$p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Diamond\varphi$$

where $p \in \mathsf{At}$.

Propositional Modal Language

A formula of Modal Logic is defined *inductively*:

1. Any element of At (called atomic propositions or propositional variables) is a formula
2. $\bot$ is a formula
3. If φ and ψ are formula, then so are $\neg\varphi$ and $\varphi \vee \psi$
4. If φ is a formula, then so is $\Diamond\varphi$
5. Nothing else is a formula

Eg., $\Box(p \rightarrow \Diamond q) \vee \Box\Diamond\neg r$; $\neg\Diamond\neg\bot$

Propositional Modal Language

The other Boolean connectives ($\wedge$, $\rightarrow$, and $\leftrightarrow$) are defined as usual

$\top$ is defined as $\neg\bot$.

$\Box\varphi$ is defined as $\neg\Diamond\neg\varphi$

$\Box p \rightarrow p$ is the formula $\neg\neg\Diamond\neg p \vee p$

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$\Diamond\varphi \ := \ \neg\Box\neg\varphi$

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where $p \in \mathrm{At}$.

Notation

- Sometimes we'll use lowercase letters $p, q, r, \ldots$ for atomic propositions and other times we'll use uppercase letters $A, B, C, \ldots$
- The choice of which modal operator is part of the syntax and which is defined is largely conventional. We will use whatever is most convenient.
- When there are multiple modal operators in the language, we will use subscripts $\Box_a$, $\Diamond_a$ or place them "inside" the operators: $[a]$, $\langle a \rangle$

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"This practice is not very consistent, but most readers should agree that it is nice to have different clothes to wear, depending on one's mood"

(van Benthem, pg. 11)

Interpreting Modal Languages: Some Warm-up Questions

1. Is $(A \to B) \vee (B \to A)$ **true** or **false**?

Interpreting Modal Languages: Some Warm-up Questions

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2. Is $A \to (B \to \neg A)$ **true** or **false**?

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7. Is $\Box A \to A$ **true** or **false**? It depends!

A few questions to keep you up at night...

- Is $A \to \Box B$ equivalent to $\Box(A \to B)$?
- Is $\Box A \to A$ valid? What about $\Box A \to \Box\Box A$?
- Can we give a truth-table semantics for the basic modal language?

  Hint: there are only 4 truth-functions for a unary operator. Suppose we want $\Box A \to A$ to be valid, but not $A \to \Box A$ and $\neg\Box A$.

Semantics for Propositional Modal Logic

1. Relational semantics (i.e., Kripke semantics)
2. Neighborhood models
3. Algebraic semantics (BAO: Boolean algebras with operators)
4. Possibility structures
5. Topological semantics (Closure algebras)
6. Category-theoretic (Coalgebras)
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Mathematical Background: sets, relations, functions, basic logic, etc.

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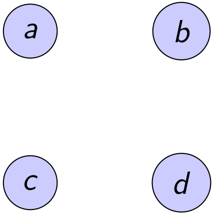

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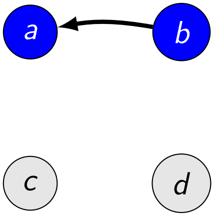

$b \; R \; a$

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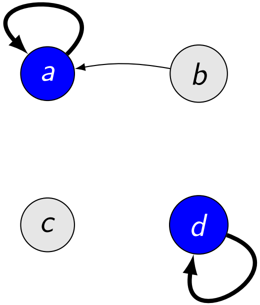

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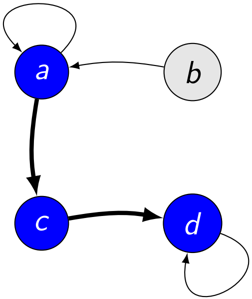

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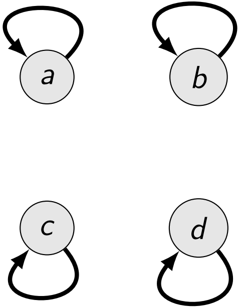

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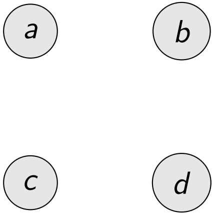

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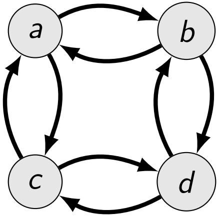

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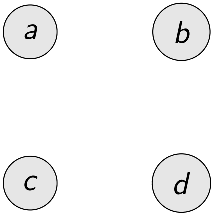

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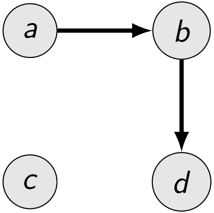

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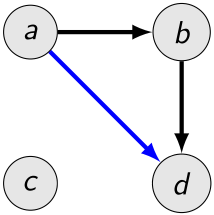

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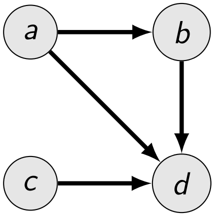

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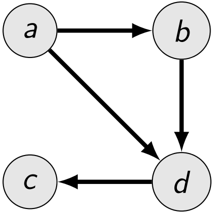

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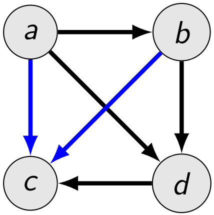

Suppose that $R \subseteq W \times W$ is a relation.

- R is *reflexive* provided that for all $w \in W$, wRw.
- R is *irreflexive* provided that for all $w \in W$, it is not the case that wRw.
- R is *symmetric* provided that for all $w, v \in W$, if wRv then vRw.
- R is *transitive* provided that for all $w, v, x \in W$, if wRv and vRx then wRx.

Suppose that $R \subseteq W \times W$ is a relation.

- R is *complete* provided that for all $w, v \in W$, wRv or vRw (or both).
- R is *serial* provided that for all $w \in W$, there is a $v \in W$ such that wRv
- R is *anti-symmetric* provided that for all $w, v \in W$, if wRv and vRw, then $w = v$.
- R is *Euclidean* provided that for all $w, v, x \in W$, if wRv and wRx then vRx.

Relational Structure

A **relational structure** is a tuple $\langle W, R\rangle$ where $W \neq \varnothing$ and $R \subseteq W \times W$ is a relation.

- Elements of the domain W are called *states*, *possible worlds*, *points*, or *nodes*.
- R is called the *accessibility relation* or the *edge relation*. When wRv we say "w can see v" or "v is accessible from w".
- For $w \in W$, let $R(w) = \{v \mid wRv\}$.

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Warning: Although a relational structure with labels is just a relational structure (with a binary relation and multiple unary relations), they have a specific interpretation in the theory of modal logic.

Examples

- Epistemic models
- Temporal models
- . . .

Muddy Children

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Assume:

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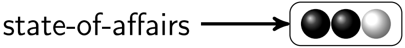

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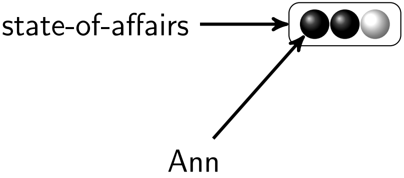

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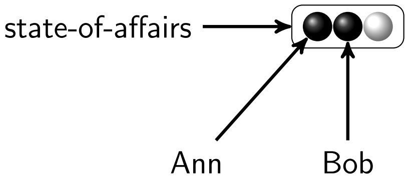

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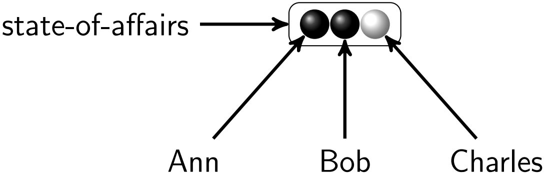

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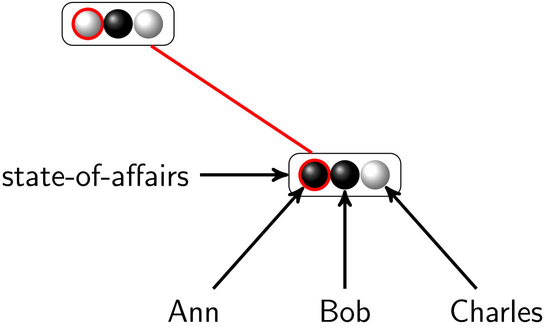

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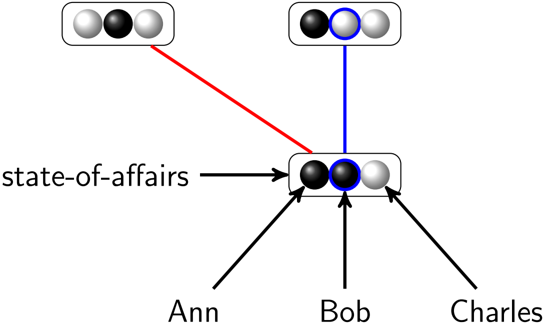

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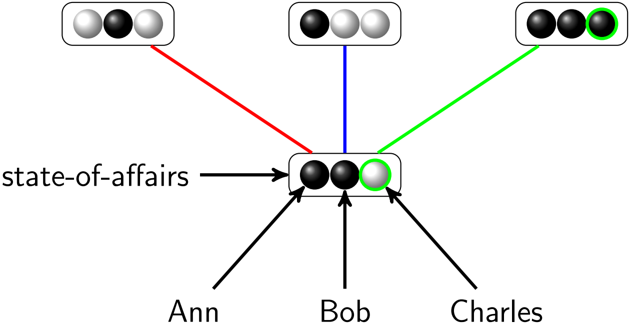

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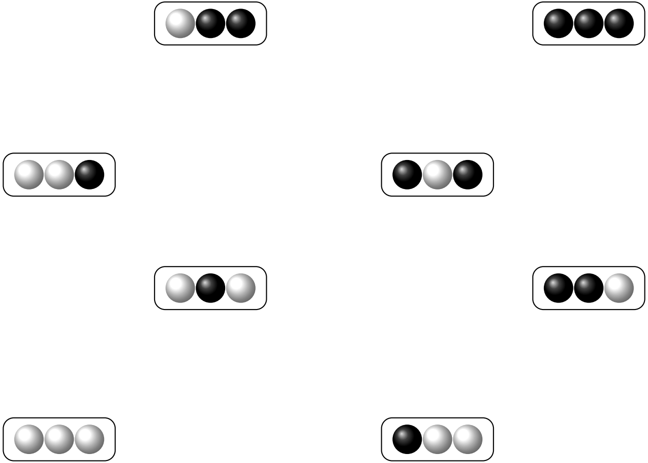
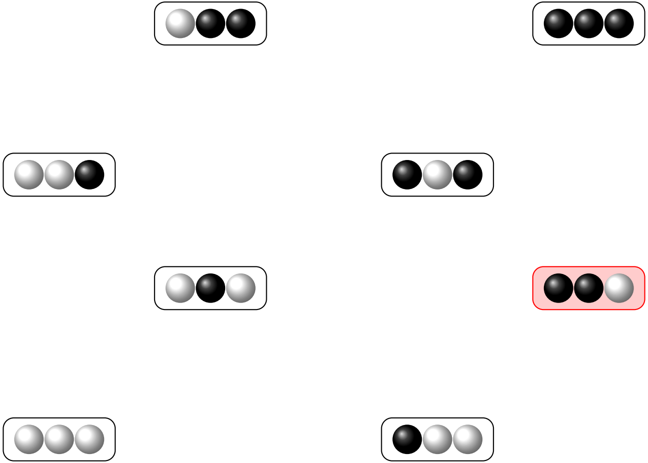


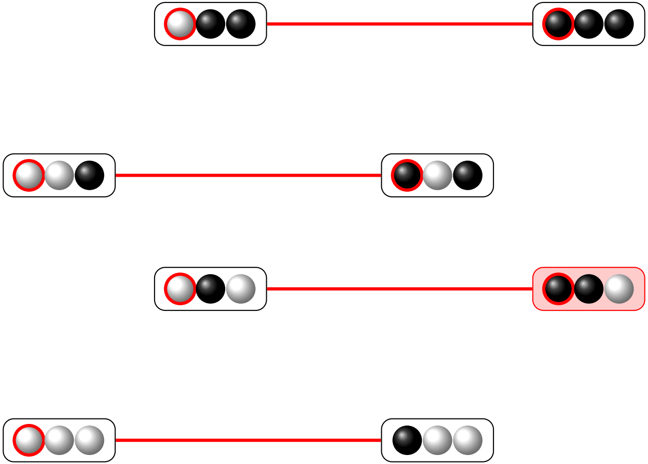
All 8 possible situations

Muddy Children



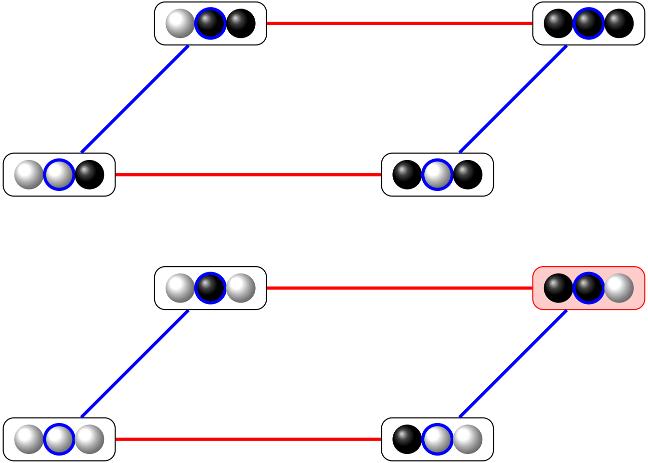
The actual situation

Muddy Children



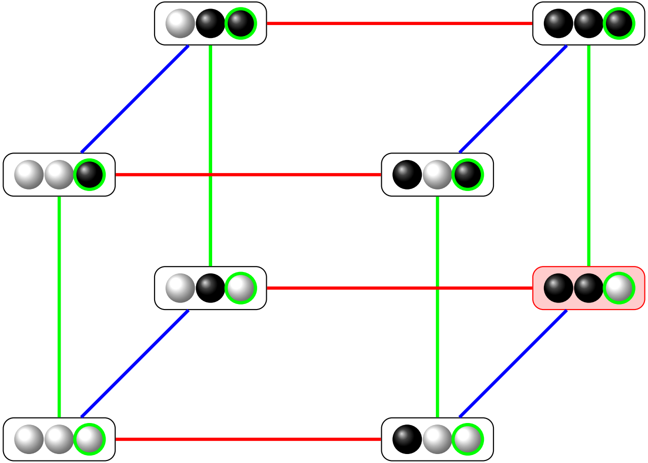
Ann's uncertainty

Muddy Children



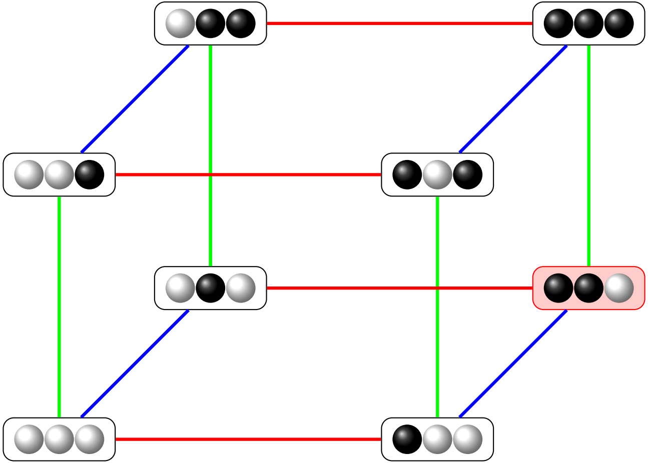
Bob's uncertainty

Muddy Children



Charles' uncertainty

Muddy Children



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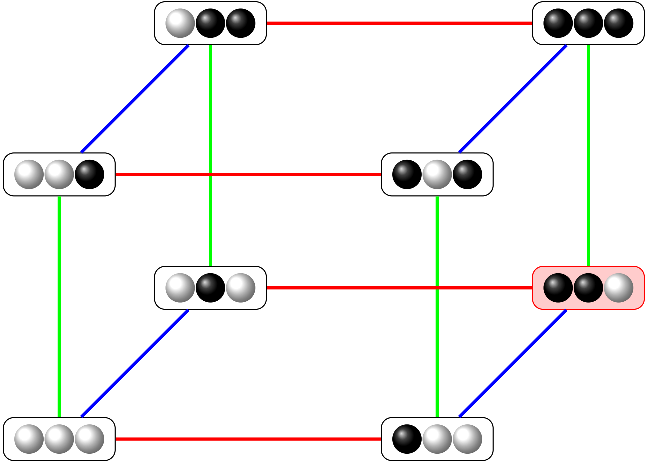
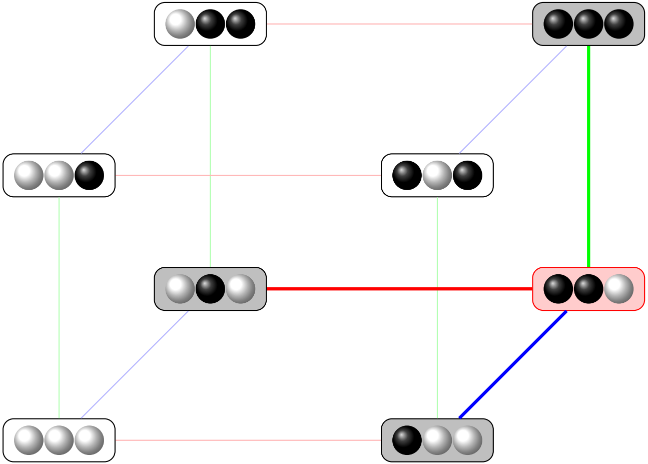


None of the children know if they are muddy

Muddy Children



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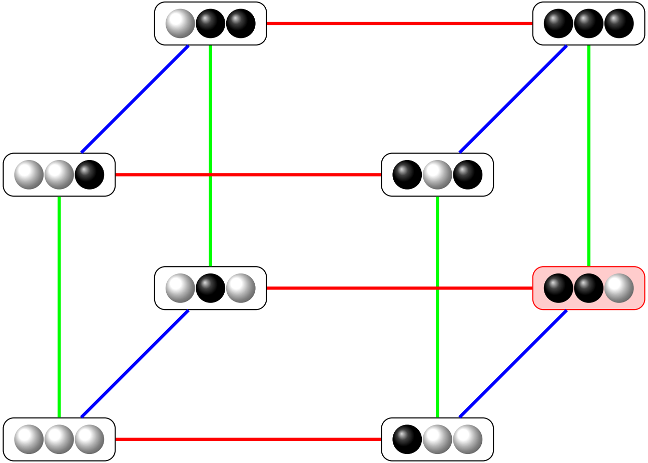
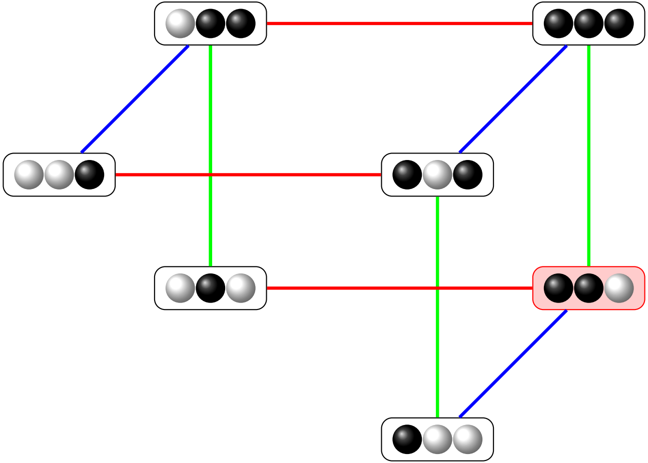


"At least one has mud on their forehead."

Muddy Children



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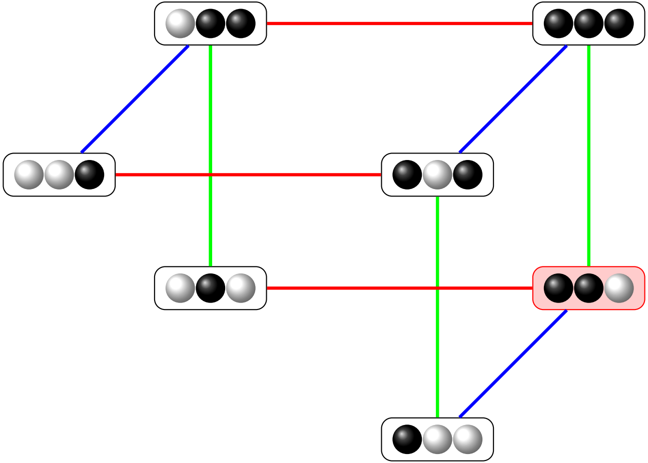

“Who has mud on their forehead?”

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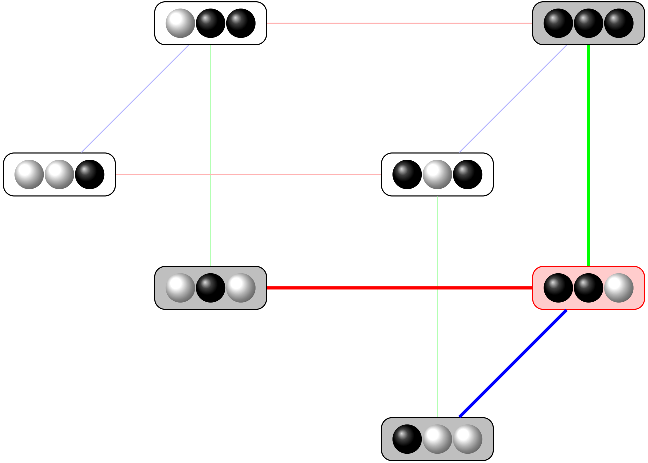

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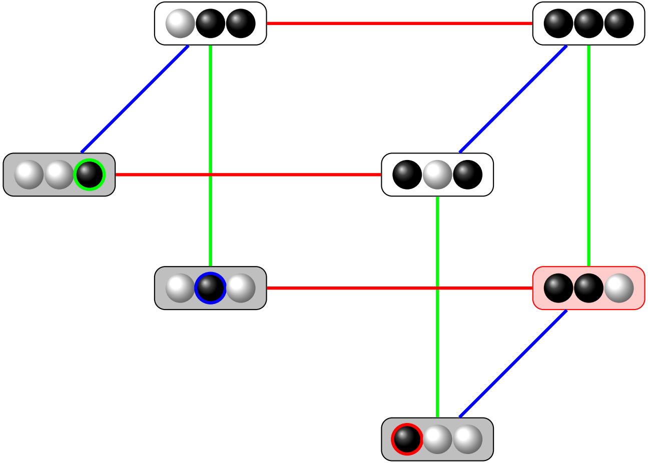
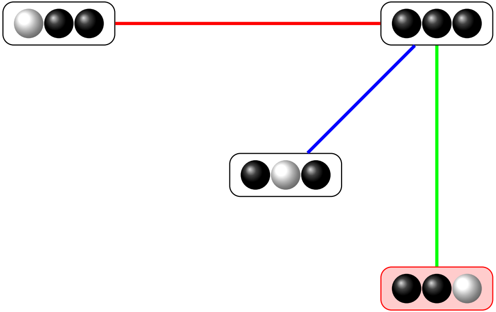


No one steps forward.

Muddy Children



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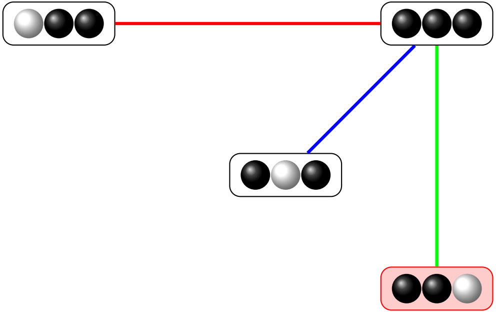

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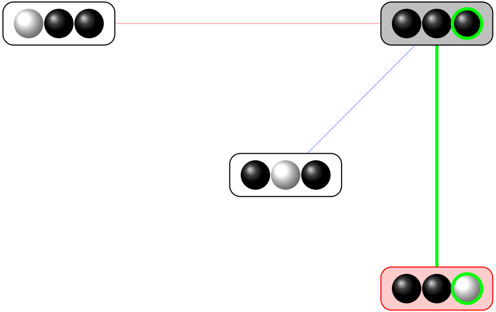
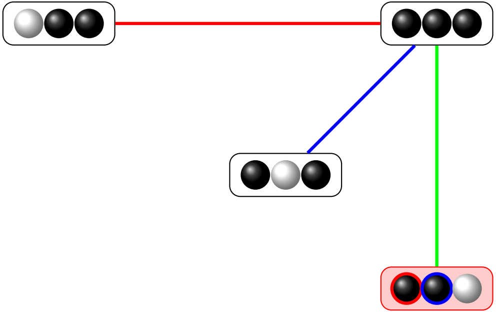


Charles does not know he is clean.

Muddy Children



Ann and Bob step forward.

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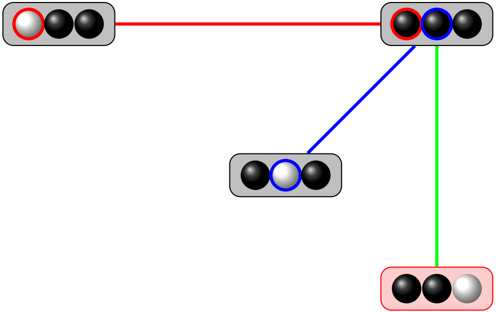

Now, Charles knows he is clean.

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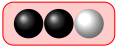

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Time

One of the most successful applications of modal logic is in the "logic of time".

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Many variations

- discrete or continuous
- branching or linear
- point based or interval based

V. Goranko and A. Galton. *Temporal Logic*. Stanford Encyclopedia of Philosophy: http://plato.stanford.edu/entries/logic-temporal/.

I. Hodkinson and M. Reynolds. *Temporal Logic*. Handbook of Modal Logic, 2008.

Models of Time

$\mathcal{T} = \langle T, < \rangle$ where

- T is a set of **time points** (or **moments**),
- $< \subseteq T \times T$ is the **precedence relation**: $s < t$ means "time point s precedes time point t (or s occurs earlier than t)" and

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Examples: $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, $\langle \mathbb{Q}, < \rangle$, $\langle \mathbb{R}, < \rangle$

Other properties of $<$

- **Linearity**: for all $s, t \in T$, $s < t$ or $s = t$ of $t < s$
- **Past-linear**: for all $s, x, y \in T$, if $x < s$ and $y < s$, then either $x < y$ or $x = y$ or $y < x$
- **Denseness** for all $s, t \in T$, if $s < t$ then there is a $z \in T$ such that $s < z$ and $z < t$
- **Discreteness**: for all $s, t \in T$, if $s < t$ then there is a z such that ($s < z$ and there is no u such that $s < u$ and $u < z$)

Branching Time

Each moment $t \in T$ can be decided into the $Past(t) = \{s \in T \mid s < t\}$ and the $Future(t) = \{s \in T \mid t < s\}$

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$F\varphi$: "it will be the case that φ"

φ will be the case "in the case in the actual course of events" or "no matter what course of events"

Branching Time Logics

A **branch** b in $\langle T, < \rangle$ is a maximal linearly ordered subset of T

$s \in T$ is **on a branch** b **of** T provided $s \in b$ (we also say "b is a branch going through t").

Temporal Logics

Temporal Logics

- *Linear Time Temporal Logic*: Reasoning about computation paths:

$F\varphi$: φ is true some time in *the* future.

A. Pnuelli. *A Temporal Logic of Programs*. in *Proc. 18th IEEE Symposium on Foundations of Computer Science* (1977).

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- *Linear Time Temporal Logic*: Reasoning about computation paths:

  $F\varphi$: φ is true some time in *the* future.

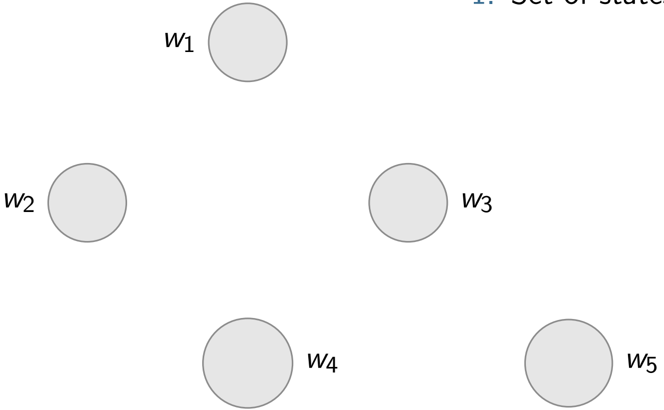
A. Pnuelli. *A Temporal Logic of Programs*. in *Proc. 18th IEEE Symposium on Foundations of Computer Science* (1977).

- *Branching Time Temporal Logic*: Allows quantification over paths:

  $\exists F\varphi$: there is a path in which φ is eventually true.

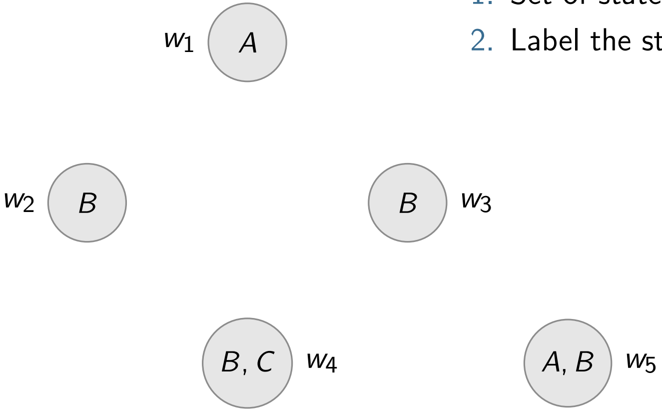
E. M. Clarke and E. A. Emerson. *Design and Synthesis of Synchronization Skeletons using Branching-time Temporal-logic Specifications*. In *Proceedings Workshop on Logic of Programs*, LNCS (1981).

Relational Model



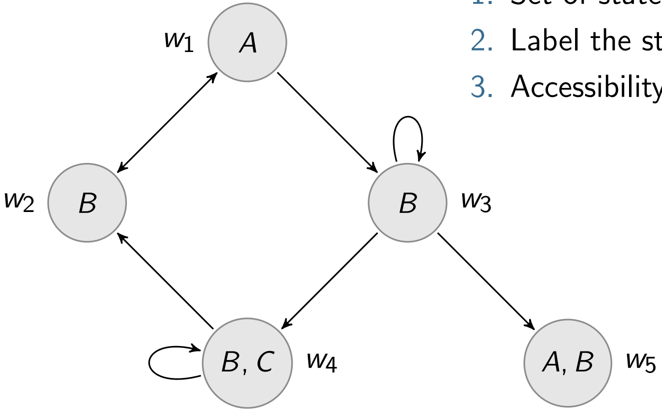
1. Set of states

Relational Model



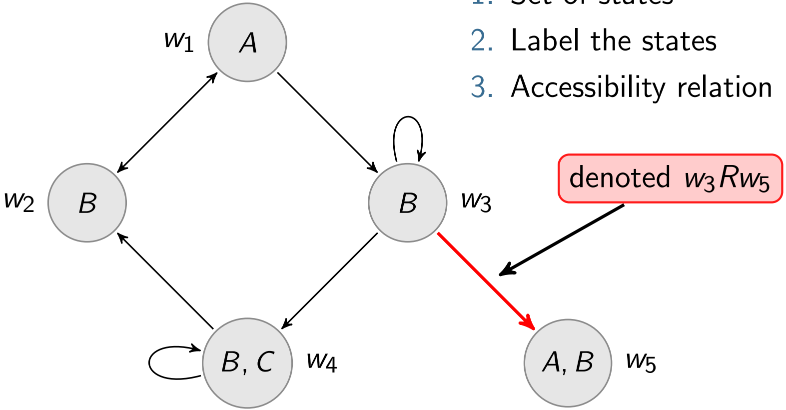
1. Set of states
2. Label the states

Relational Model



1. Set of states
2. Label the states
3. Accessibility relation

Relational Model



1. Set of states
2. Label the states
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Frame: $\langle W, R \rangle$, where $W \neq \varnothing$ and $R \subseteq W \times W$

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Model: Suppose that $\mathcal{F} = \langle W, R \rangle$ is a frame. The tuple $\langle W, R, V \rangle$ is a **model based on** $\mathcal{F}$ where $V : \mathrm{At} \to \wp(W)$ is a **valuation function**.

- $w \in V(p)$ means that p is true at w.

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Pointed Model Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. If $w \in W$, then $(\mathcal{M}, w)$ is called a **pointed model**.

Truth of Modal Formulas

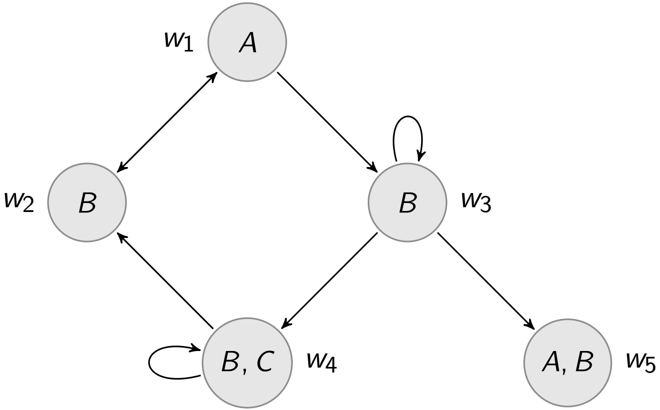
Suppose that $\mathcal{M} = \langle W, R, V \rangle$ is a model. Truth of a modal formula $\varphi \in \mathcal{L}(\mathsf{At})$ at a state w in $\mathcal{M}$, denoted $\mathcal{M}, w \models \varphi$, is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (where $p \in \mathsf{At}$)
- $\mathcal{M}, w \not\models \bot$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

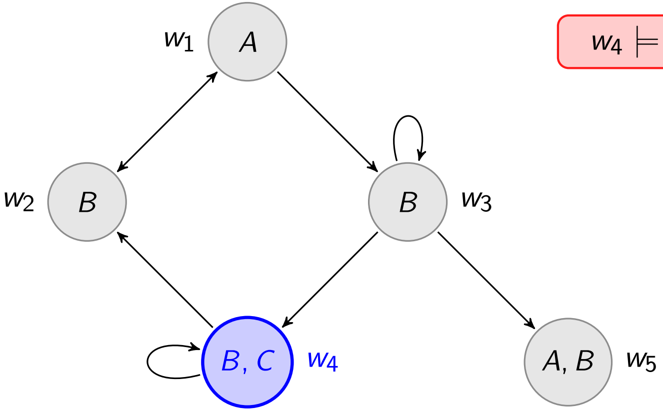
Truth of Modal Formulas

- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \rightarrow \psi$ iff if $\mathcal{M}, w \models \varphi$, then $\mathcal{M}, w \models \psi$
  iff either $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\varphi$ iff for all $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$

Example

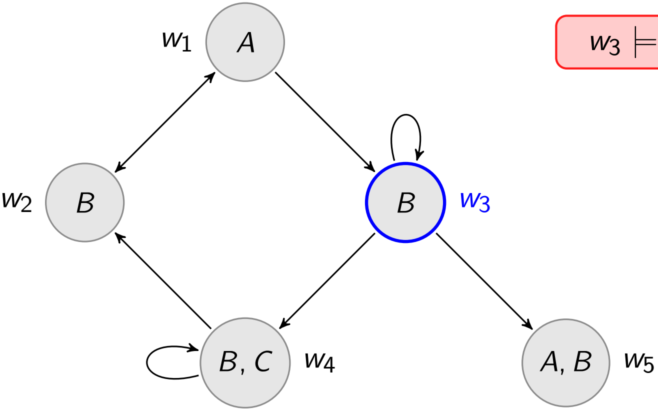


Example



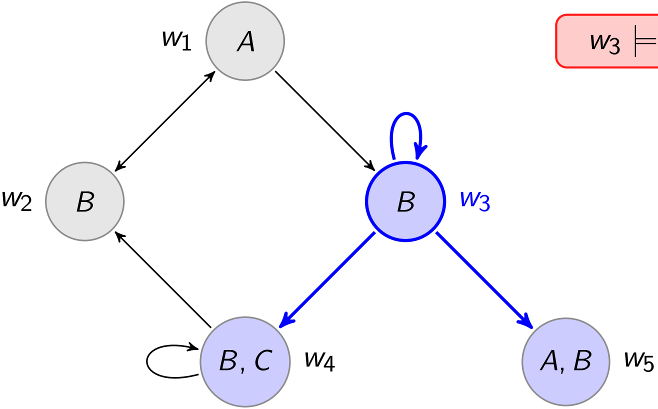
$w_4 \models B \wedge C$

Example



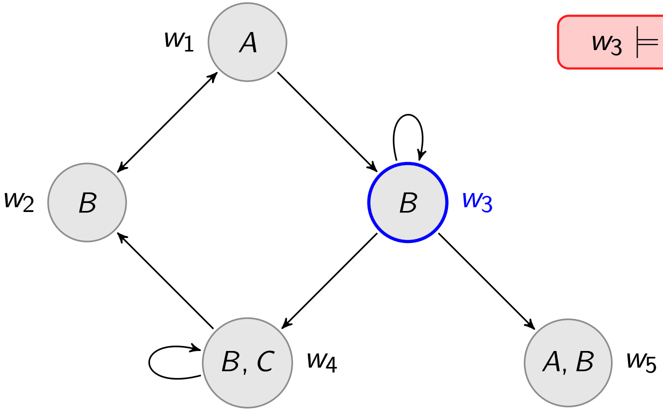
$w_3 \models \Box B$

Example



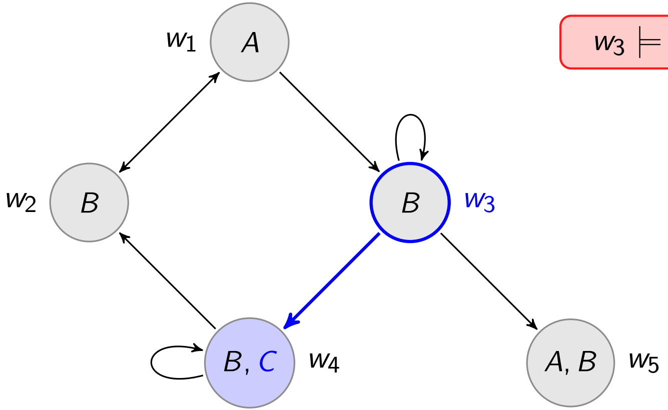
$w_3 \models \Box B$

Example



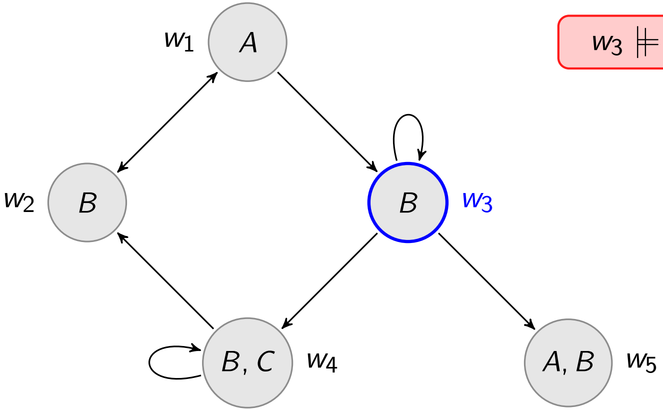
$w_3 \models \Diamond C$

Example



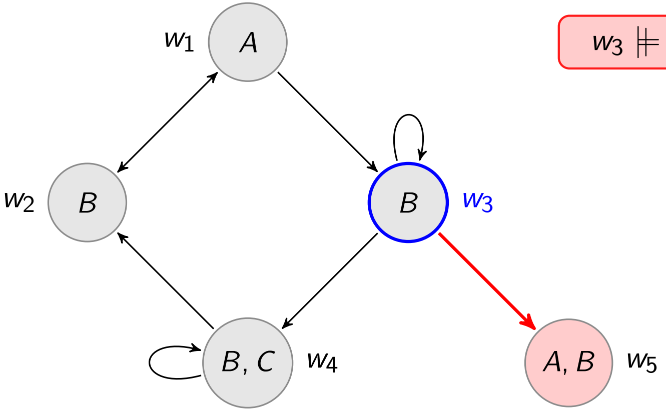
$w_3 \models \Diamond C$

Example



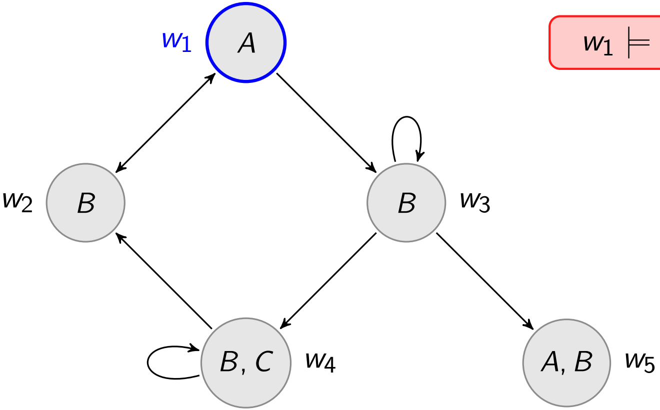
$$\boxed{w_3 \not\models \Box C}$$

Example



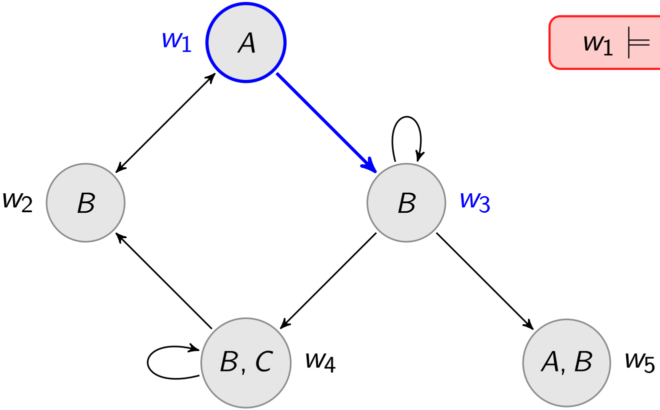
$w_3 \not\models \Box C$

Example



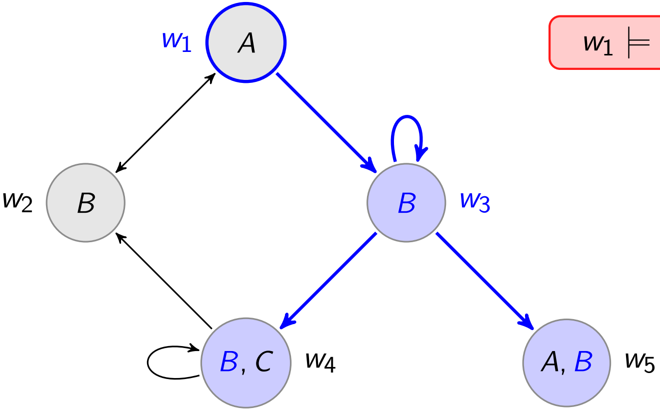
$w_1 \models \Diamond\Box B$

Example



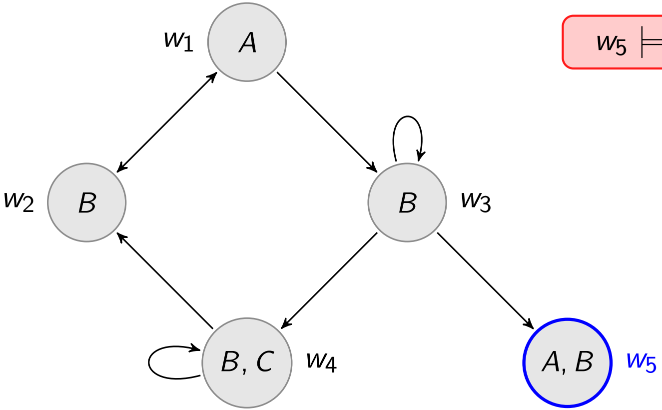
$w_1 \models \Diamond\Box B$

Example



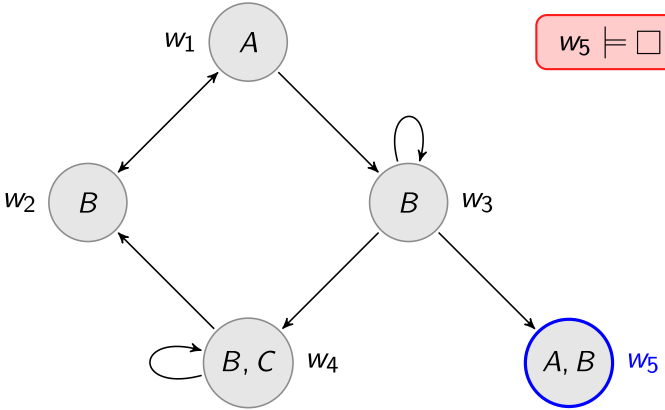
$w_1 \models \Diamond\Box B$

Example



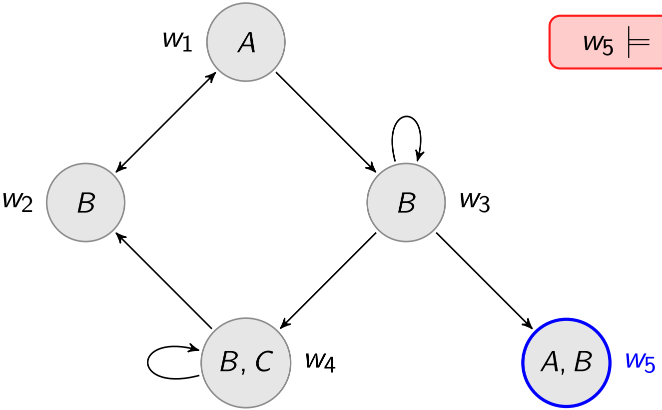
$w_5 \models \Box C$

Example

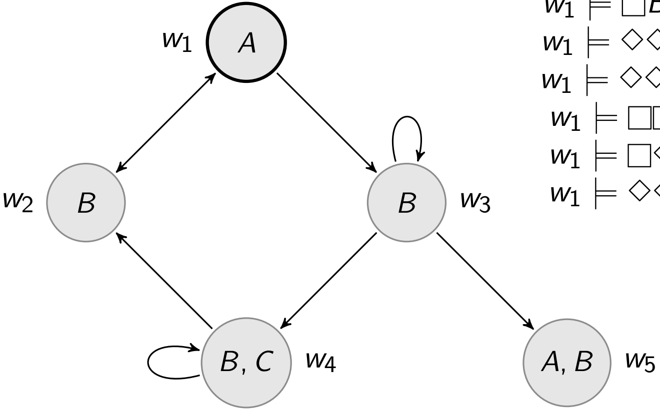


$w_5 \models \Box(B \wedge \neg B)$

Example



$w_5 \models \neg\Diamond B$



$w_1 \models \Box B \wedge B?$

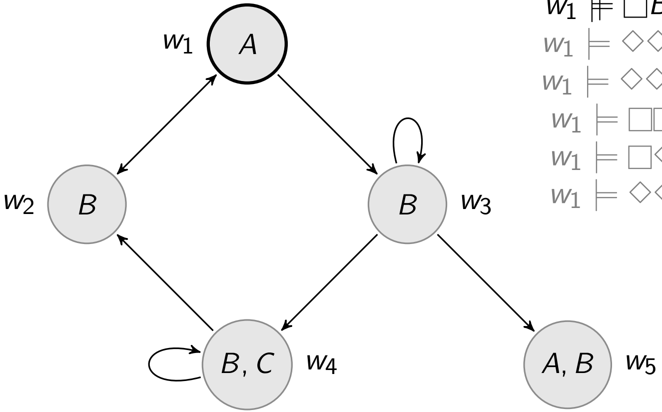
$w_1 \models \Diamond\Diamond B?$

$w_1 \models \Diamond\Diamond\Diamond B?$

$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

$w_1 \models \Diamond\Diamond B?$

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$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

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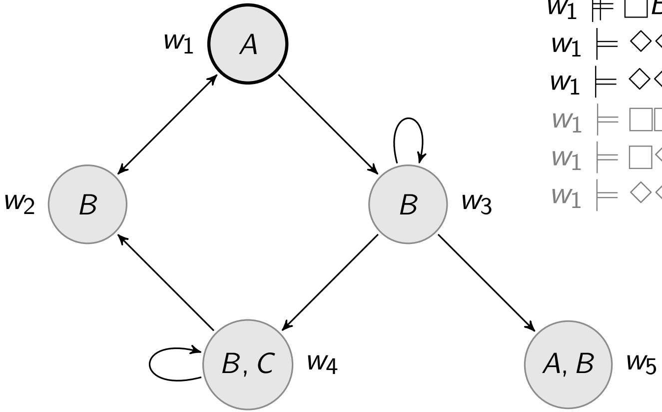

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$w_1 \models \Diamond\Diamond B$

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$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

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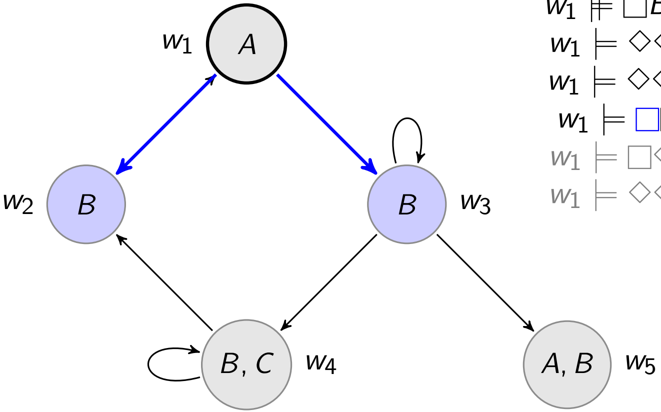

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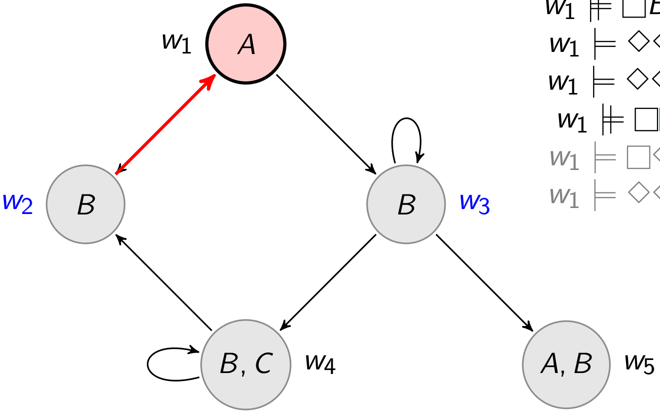

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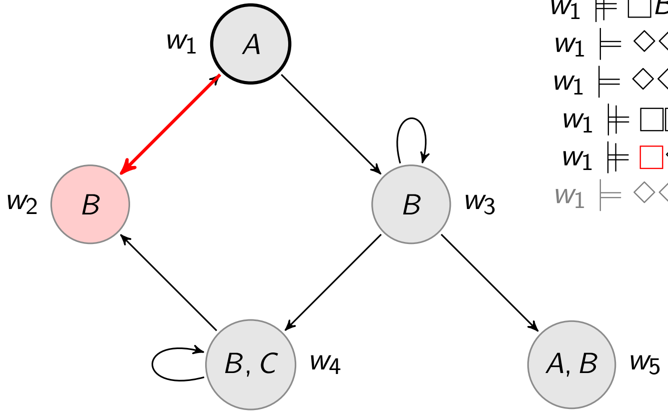

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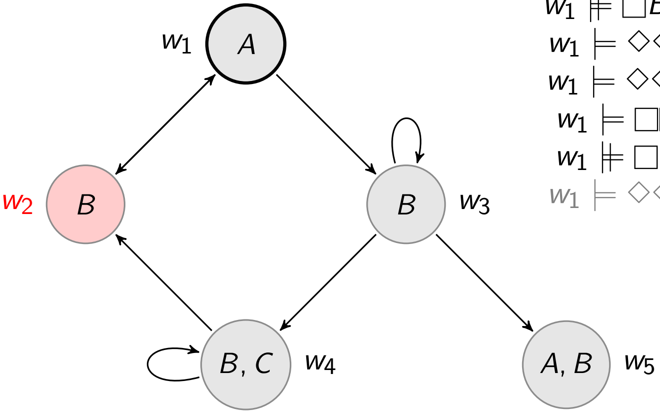

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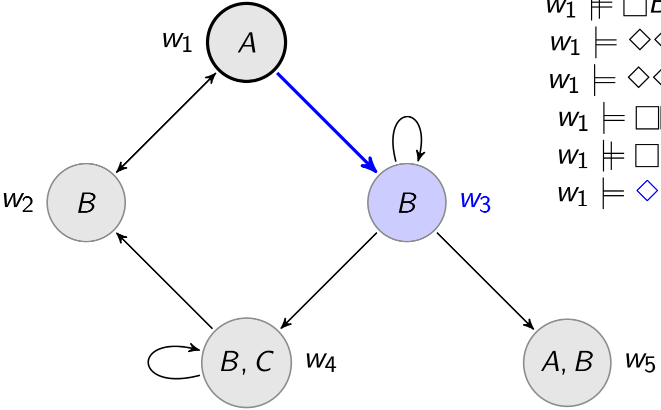

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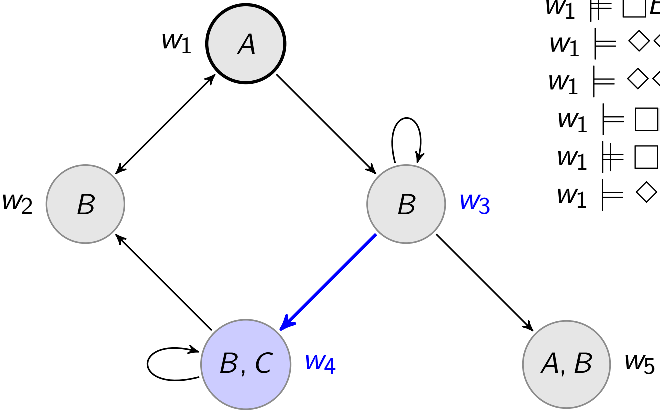

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$w_1 \not\models \Box\Diamond C$

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φ is **satisfiable** means that there is a model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$ such that $\mathcal{M}, w \models \varphi$.

Validity

Valid on a model $\mathcal{M} = \langle W, V, R \rangle$

$\mathcal{M} \models \varphi$: for all $w \in W$, $\mathcal{M}, w \models \varphi$

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for all functions V, for all $w \in W$, $\langle W, R, V \rangle, w \models \varphi$

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for all functions V, for all $w \in W$, $\langle W, R, V \rangle, w \models \varphi$

Valid at a state on a frame $\mathcal{F} = \langle W, R \rangle$ with $w \in W$

$\mathcal{F}, w \models \varphi$: for all $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M}, w \models \varphi$

Validity

Valid on a model $\mathcal{M} = \langle W, V, R \rangle$

$\mathcal{M} \models \varphi$: for all $w \in W$, $\mathcal{M}, w \models \varphi$

Valid on a frame $\mathcal{F} = \langle W, R \rangle$

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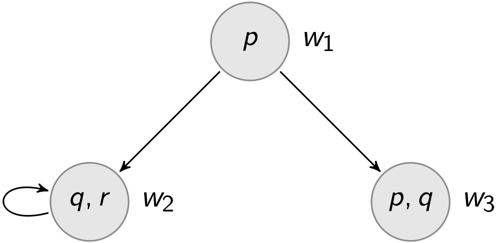
Valid at a state on a frame $\mathcal{F} = \langle W, R \rangle$ with $w \in W$

$\mathcal{F}, w \models \varphi$: for all $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M}, w \models \varphi$

Valid in a class F of frames:

$\models_{\mathsf{F}} \varphi$: for all $\mathcal{F} \in \mathsf{F}$, $\mathcal{F} \models \varphi$

Model validity



$$\mathcal{M} \models \Box q$$

validity on a model is *not* closed under substitution ($\mathcal{M} \not\models \Box p$)